

FINITENESS PROPERTIES OF LOCAL COHOMOLOGY FOR F -PURE LOCAL RINGS

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ABSTRACT. In this paper, we show that for an F -pure local ring (R, \mathfrak{m}) , all local cohomology modules $H_{\mathfrak{m}}^i(R)$ have finitely many Frobenius compatible submodules. This answers positively the open question raised by F. Enescu and M. Hochster in [EH08] (see also [Ene12], where it was stated as a conjecture when R is Cohen-Macaulay). We also prove that when (R, \mathfrak{m}) is excellent and F -pure on the punctured spectrum, all local cohomology modules $H_{\mathfrak{m}}^i(R)$ have finite length in the category of R -modules with Frobenius action. Finally, we show that the property that all $H_{\mathfrak{m}}^i(R)$ have finitely many Frobenius compatible submodules passes to localizations.

1. INTRODUCTION

Let (R, \mathfrak{m}) be a local ring of equal characteristic $p > 0$ of dimension d . There is a natural action of the Frobenius endomorphism of R on each of its local cohomology modules $H_{\mathfrak{m}}^i(R)$. We call an R -submodule N of $H_{\mathfrak{m}}^i(R)$ *F-compatible* if F maps N into itself. Our main interest is to understand when a local ring (R, \mathfrak{m}) has the property that there are only finitely many F -compatible submodules for each $H_{\mathfrak{m}}^i(R)$, $0 \leq i \leq d$. Rings with this property are called *FH-finite* and have been studied in detail in [EH08]. It was proved in [EH08] that an F -pure Gorenstein ring is FH-finite. This also follows from results in [Sha07]. Recently, it was proved in [Ene12] that if an F -injective Cohen-Macaulay ring R admits a canonical ideal $I \cong \omega_R$ such that R/I is F -pure, then R is FH-finite. We notice that all these hypothesis imply R is F -pure. In fact, it was asked in [EH08] (and this was later conjectured in [Ene12] for Cohen-Macaulay rings) whether the F -pure property itself is enough for FH-finiteness. We provide a positive answer to this question. We actually prove a stronger result, which says that F -pure will imply *stably FH-finite*, this means that R and all power series rings over R are FH-finite. Our first main result is the following, proved in Section 3:

Theorem 1.1. *Let (R, \mathfrak{m}) be an F -pure local ring. Then R and all power series rings over R are FH-finite, that is, R is stably FH-finite.*

We shall also study the problem of determining conditions under which the local cohomology modules $H_{\mathfrak{m}}^i(R)$ have finite length in the category of R -modules with Frobenius action. In fact, rings with this property are said to have *FH-finite length* and have been studied in [EH08]. Inspired by the theory of *Cartier modules* introduced by M. Blickle and G. Böklen in [BB11] and the Γ -construction introduced by M. Hochster and C. Huneke in [HH94a], we obtain our second and third main result, proved in Section 5:

Theorem 1.2. *Let (R, \mathfrak{m}) be an excellent local ring such that R_P is F -pure for every $P \in \text{Spec } R - \{\mathfrak{m}\}$. Then R has FH-finite length.*

Theorem 1.3. *Let (R, \mathfrak{m}) be a local ring that has FH-finite length (resp. is FH-finite or stably FH-finite). Then so is R_P for every $P \in \operatorname{Spec} R$.*

Throughout this paper we always assume (R, \mathfrak{m}) is a Noetherian local ring of equal characteristic $p > 0$, except at the beginning of Section 3, where the hypothesis will be stated clearly. In Section 2 we start with some definitions and properties of F -pure and F -split rings, and we recall some theorems on the FH-finiteness and anti-nilpotency of $H_{\mathfrak{m}}^i(R)$ proved in [EH08]. In Section 3 we prove Theorem 1.1. In Section 4 we prove a key theorem, Theorem 4.7 which implies both Theorem 1.2 and Theorem 1.3 when R is complete and F -finite. In Section 5, we make use of the Γ -construction in [HH94a] to prove Theorem 1.2 and Theorem 1.3 in full generality. Finally, in Section 6, we study some examples. Some of our techniques are inspired by the work in [SW07].

2. FH-FINITENESS AND ANTI-NILPOTENCY

Recall that a map of R -modules $N \rightarrow N'$ is *pure* if for every R -module M the map $N \otimes_R M \rightarrow N' \otimes_R M$ is injective. This implies that $N \rightarrow N'$ is injective, and is weaker than the condition that $0 \rightarrow N \rightarrow N'$ be split. A local ring (R, \mathfrak{m}) is called *F -pure* (respectively, *F -split*) if the Frobenius endomorphism $F: R \rightarrow R$ is pure (respectively, split). Evidently, an F -split ring is F -pure and an F -pure ring is reduced. When R is either F -finite or complete, F -pure and F -split are equivalent (cf. Discussion 2.6 in [EH08]).

We will use some notations introduced in [EH08]. We say an R -module M is an $R\{F\}$ -module if there is a Frobenius action $F: M \rightarrow M$ such that for all $u \in M$, $F(ru) = r^p u$. This is same as saying that M is a left module over the ring $R\{F\}$, which may be viewed as a noncommutative ring generated over R by the symbols $1, F, F^2, \dots$ by requiring that $Fr = r^p F$ for $r \in R$. We say N is an *F -compatible* submodule of M if $F(N) \subseteq N$. Note that this is equivalent to requiring that N be an $R\{F\}$ -submodule of M . We say an $R\{F\}$ -module M is *F -nilpotent* if $F^e(M) = 0$ for some e .

The Frobenius endomorphism on R induces a natural Frobenius action on each local cohomology module $H_{\mathfrak{m}}^i(R)$ (see Discussion 2.2 and 2.4 in [EH08] for a detailed explanation of this). We say a local ring is *F -injective* if F acts injectively on all of the local cohomology modules of R with support in \mathfrak{m} . This holds if R is F -pure.

Definition 2.1 (cf. Definition 2.5 in [EH08]). A local ring (R, \mathfrak{m}) of dimension d is called *FH-finite* if for all $0 \leq i \leq d$, there are only finitely many F -compatible submodules of $H_{\mathfrak{m}}^i(R)$. We say (R, \mathfrak{m}) has *FH-finite length* if for each $0 \leq i \leq d$, $H_{\mathfrak{m}}^i(R)$ has finite length in the category of $R\{F\}$ -modules.

In [EH08], F. Enescu and M. Hochster also introduced the anti-nilpotency condition for $R\{F\}$ -modules, which turns out to be very useful and has close connections with the finiteness properties of local cohomology modules. In fact, it is proved in [EH08] that the anti-nilpotency of $H_{\mathfrak{m}}^i(R)$ for all i is equivalent to the condition that all power series rings over R be FH-finite.

Definition 2.2. Let (R, \mathfrak{m}) be a local ring and let W be an $R\{F\}$ -module. We say W is *anti-nilpotent* if for every F -compatible submodule $V \subseteq W$, F acts injectively on W/V .

Theorem 2.3 (cf. Theorem 4.15 in [EH08]). *Let (R, \mathfrak{m}) be a local ring and let x_1, \dots, x_n be formal power series indeterminates over R . Let $R_0 = R$ and $R_n = R[[x_1, \dots, x_n]]$. Then the following conditions on R are equivalent:*

- (1) All local cohomology modules $H_{\mathfrak{m}}^i(R)$ are anti-nilpotent.
- (2) The ring R_n is FH-finite for every n .
- (3) $R_1 \cong R[[x]]$ has FH-finite length.

When R satisfies these equivalent conditions, we call it stably FH-finite.

We will also need some results in [BB11] about Cartier modules. We recall some definitions in [BB11]. The definitions and results in [BB11] work for schemes and sheaves, but we will only give the corresponding definitions for local rings for simplicity (we will not use the results on schemes and sheaves).

Definition 2.4. A *Cartier module* over R is an R -module equipped with a p^{-1} linear map $C_M: M \rightarrow M$, that is an additive map satisfying $C(r^p x) = rC(x)$ for every $r \in R$ and $x \in M$. A Cartier module (M, C) is called *nilpotent* if $C^e(M) = 0$ for some e .

Remark 2.5. (1) A Cartier module is nothing but a right module over the ring $R\{F\}$.
(2) If (M, C) is a Cartier module, then $C_P: M_P \rightarrow M_P$ defined by

$$C_P\left(\frac{x}{r}\right) = \frac{C(r^{p-1}x)}{r}$$

for every $x \in M$ and $r \in R - P$ gives M_P a Cartier module structure over R_P .

We end this section with a simple lemma which will reduce most problems to the F -split case (recall that for complete local rings, F -pure is equivalent to F -split).

Lemma 2.6. *Let (R, \mathfrak{m}) be a local ring. Then R has FH-finite length (resp. is FH-finite or stably FH-finite) if and only if \hat{R} has FH-finite length (resp. is FH-finite or stably FH-finite).*

Proof. Since completion does not affect either what the local cohomology modules are nor what the action of Frobenius is, the lemma follows immediately. \square

3. F -PURE IMPLIES STABLY FH-FINITE

In order to prove the main result, we begin with some simple Lemmas 3.1, 3.2, 3.3 and a Proposition 3.4 which are characteristic free. In fact, in all these lemmas we only need to assume I is a finitely generated ideal so that the Čech complex characterization of local cohomology can be applied.

Lemma 3.1. *Let R be a Noetherian ring, I be an ideal of R and M be any R -module. We have a natural map:*

$$M \otimes_R H_I^i(R) \xrightarrow{\phi} H_I^i(M)$$

Moreover, when $M = S$ is an R -algebra, ϕ is S -linear.

Proof. Given maps of R -modules $L_1 \xrightarrow{\alpha} L_2 \xrightarrow{\beta} L_3$ and $M \otimes_R L_1 \xrightarrow{id \otimes \alpha} M \otimes_R L_2 \xrightarrow{id \otimes \beta} M \otimes_R L_3$ such that $\beta \circ \alpha = 0$, there is a natural map:

$$M \otimes_R \frac{\ker \beta}{\operatorname{im} \alpha} \rightarrow \frac{\ker(id \otimes \beta)}{\operatorname{im}(id \otimes \alpha)}$$

sending $m \otimes \bar{z}$ to $\overline{m \otimes z}$. Now the result follows immediately by the Čech complex characterization of local cohomology. \square

Lemma 3.2. *Let R be a Noetherian ring, S be an R -algebra, and I be an ideal of R . We have a commutative diagram:*

$$\begin{array}{ccc} & S \otimes_R H_I^i(R) & \\ j_2 \nearrow & \downarrow \phi & \\ H_I^i(R) & \xrightarrow{j_1} & H_{IS}^i(S) \end{array}$$

where j_1, j_2 are the natural maps induced by $R \rightarrow S$, in particular j_2 sends z to $1 \otimes z$.

Proof. This is straightforward to check. \square

Lemma 3.3. *Let R be a Noetherian ring and S be an R -algebra such that the inclusion $\iota: R \hookrightarrow S$ splits. Let γ be the splitting $S \rightarrow R$. Then we have a commutative diagram:*

$$\begin{array}{ccc} & S \otimes_R H_I^i(R) & \\ q_2 \nwarrow & \downarrow \phi & \\ H_I^i(R) & \xleftarrow{q_1} & H_{IS}^i(S) \end{array}$$

where q_1, q_2 are induced by γ , in particular q_2 sends $s \otimes z$ to $\gamma(s)z$.

Proof. We may identify S with $R \oplus W$ and $R \hookrightarrow S$ with $R \hookrightarrow R \oplus W$ which sends r to $(r, 0)$, and $S \rightarrow R$ with $R \oplus W \rightarrow R$ which sends (r, w) to r (we may take W to be the R -submodule of S generated by $s - \iota \circ \gamma(s)$). Under this identification, we have:

$$S \otimes_R H_I^i(R) = H_I^i(R) \oplus W \otimes_R H_I^i(R)$$

$$H_{IS}^i(S) = H_I^i(R) \oplus H_I^i(W)$$

and q_1, q_2 are just the projections onto the first factors. Now the conclusion is clear because by Lemma 3.1, $\phi: S \otimes_R H_I^i(R) \rightarrow H_{IS}^i(S)$ is the identity on $H_I^i(R)$ and sends $W \otimes_R H_I^i(R)$ to $H_I^i(W)$. \square

Proposition 3.4. *Let R be a Noetherian ring and S be an R -algebra such that $R \hookrightarrow S$ splits. Let y be an element in $H_I^i(R)$ and N be a submodule of $H_I^i(R)$. If the image of y is in the S -span of the image of N in $H_{IS}^i(S)$, then $y \in N$.*

Proof. We know there are two commutative diagrams as in Lemma 3.2 and 3.3 (note that here j_1 and j_2 are inclusions since $R \hookrightarrow S$ splits). We use γ to denote the splitting $S \rightarrow R$. The condition says that $j_1(y) = \sum s_k \cdot j_1(n_k)$ for some $s_k \in S$ and $n_k \in N$. Applying q_1 we get:

$$\begin{aligned} y &= q_1 \circ j_1(y) \\ &= \sum q_1(s_k \cdot j_1(n_k)) \\ &= \sum q_1(s_k \cdot \phi \circ j_2(n_k)) \\ &= \sum q_1 \circ \phi(s_k \cdot j_2(n_k)) \\ &= \sum q_2(s_k \otimes n_k) \\ &= \sum \gamma(s_k) \cdot n_k \in N \end{aligned}$$

where the first equality is by definition of q_1 , the third equality is by Lemma 3.2, the fourth equality is because ϕ is S -linear, the fifth equality is by Lemma 3.3 and the definition of j_2 and the last equality is by the definition of q_2 . This finishes the proof. \square

Now we return to the situation in which we are interested. We assume (R, \mathfrak{m}) is a Noetherian local ring of equal characteristic $p > 0$. We first prove an immediate corollary of Proposition 3.4, which explains how FH-finite and stably FH-finite properties behave under split maps.

Corollary 3.5. *Suppose $(R, \mathfrak{m}) \hookrightarrow (S, \mathfrak{n})$ is split and $\mathfrak{m}S$ is primary to \mathfrak{n} . Then if S is FH-finite (respectively, stably FH-finite), so is R .*

Proof. First notice that, when $R \hookrightarrow S$ is split, so is $R[[x_1, \dots, x_n]] \hookrightarrow S[[x_1, \dots, x_n]]$. So it suffices to prove the statement for FH-finite. Since $\mathfrak{m}S$ is primary to \mathfrak{n} , for every i , we have a natural commutative diagram:

$$\begin{array}{ccc} H_{\mathfrak{m}}^i(R) & \longrightarrow & H_{\mathfrak{n}}^i(S) \\ \downarrow F & & \downarrow F \\ H_{\mathfrak{m}}^i(R) & \longrightarrow & H_{\mathfrak{n}}^i(S) \end{array}$$

where the horizontal maps are induced by the inclusion $R \hookrightarrow S$, and the vertical maps are the Frobenius action. It is straightforward to check that if N is an F -compatible submodule of $H_{\mathfrak{m}}^i(R)$, then the S -span of N is also an F -compatible submodule of $H_{\mathfrak{n}}^i(S)$.

If N_1 and N_2 are two different F -compatible submodules of $H_{\mathfrak{m}}^i(R)$, then their S -span in $H_{\mathfrak{n}}^i(S)$ must be different by Proposition 3.4. But since S is FH-finite, each $H_{\mathfrak{n}}^i(S)$ only has finitely many F -compatible submodules. Hence so is $H_{\mathfrak{m}}^i(R)$. This finishes the proof. \square

Now we start proving our main result. First we prove a lemma:

Lemma 3.6. *Let W be an $R\{F\}$ -module. Then W is anti-nilpotent if and only if for every $y \in W$, $y \in \text{span}_R\langle F(y), F^2(y), F^3(y), \dots \rangle$.*

Proof. Suppose W is anti-nilpotent. For each $y \in W$, $V := \text{span}_R\langle F(y), F^2(y), F^3(y), \dots \rangle$ is an F -compatible submodule of W . Hence, F acts injectively on W/V by anti-nilpotency of W . But clearly $F(\bar{y}) = 0$ in W/V , so $\bar{y} = 0$, so $y \in V$.

For the other direction, suppose there exists some F -compatible submodule $V \subseteq W$ such that F does not act injectively on W/V . We can pick some $y \notin V$ such that $F(y) \in V$. Since V is an F -compatible submodule and $F(y) \in V$, $\text{span}_R\langle F(y), F^2(y), F^3(y), \dots \rangle \subseteq V$. So $y \in \text{span}_R\langle F(y), F^2(y), F^3(y), \dots \rangle \subseteq V$ which is a contradiction. \square

Theorem 3.7. *Let (R, \mathfrak{m}) be a local ring which is F -split. Then $H_{\mathfrak{m}}^i(R)$ is anti-nilpotent for every i .*

Proof. By Lemma 3.6, it suffices to show for every $y \in H_{\mathfrak{m}}^i(R)$, we have

$$y \in \text{span}_R\langle F(y), F^2(y), F^3(y), \dots \rangle.$$

Let $N_j = \text{span}_R\langle F^j(y), F^{j+1}(y), \dots \rangle$, consider the descending chain:

$$N_0 \supseteq N_1 \supseteq N_2 \supseteq \dots \supseteq N_j \supseteq \dots$$

Since $H_{\mathfrak{m}}^i(R)$ is Artinian, this chain stabilizes, so there exists a smallest e such that $N_e = N_{e+1}$. If $e = 0$ we are done. Otherwise we have $F^{e-1}(y) \notin N_e$. Since R is F -split, we apply

Proposition 3.4 to the Frobenius map $R \xrightarrow{r \mapsto r^p} R = S$ (and $I = \mathfrak{m}$). In order to make things clear we use S to denote the target R , but we keep in mind that $S = R$.

From Proposition 3.4 we know that the image of $F^{e-1}(y)$ is not contained in the S -span of the image of N_e under the map $H_{\mathfrak{m}}^i(R) \rightarrow H_{\mathfrak{m}S}^i(S) \cong H_{\mathfrak{m}}^i(R)$. But this map is exactly the Frobenius map on $H_{\mathfrak{m}}^i(R)$, so the image of $F^{e-1}(y)$ is $F^e(y)$, and after identifying S with R , the S -span of the image of N_e is the R -span of $F^{e+1}(y), F^{e+2}(y), F^{e+3}(y), \dots$ which is N_{e+1} . So $F^e(y) \notin N_{e+1}$, which contradicts our choice of e . \square

Theorem 3.8. *Let (R, \mathfrak{m}) be an F -pure local ring. Then R and all power series rings over R are FH-finite (i.e. R is stably FH-finite).*

Proof. By Lemma 2.6, we may assume R is F -split. Now the result is clear from Theorem 3.7 and Theorem 2.3. \square

4. FH-FINITE LENGTH AND STABLE FH-FINITENESS IN THE COMPLETE F -FINITE CASE

In this section, we show for a complete and F -finite local ring (R, \mathfrak{m}) , the condition that R_P be stably FH-finite for all $P \in \text{Spec } R - \{\mathfrak{m}\}$ is equivalent to the condition that R have FH-finite length. First we recall the following important theorem of Lyubeznik:

Theorem 4.1 (cf. Theorem 4.7 in [Lyu97] or Theorem 4.7 in [EH08]). *Let W be an $R\{F\}$ -module which is Artinian as an R -module. Then W has a finite filtration*

$$0 = L_0 \subseteq N_0 \subseteq L_1 \subseteq N_1 \subseteq \dots \subseteq L_s \subseteq N_s = W$$

by Frobenius compatible submodules of W such that every N_j/L_j is F -nilpotent, while every L_j/N_{j-1} is simple in the category of $R\{F\}$ -modules, with a nonzero Frobenius action. The integer s and the isomorphism classes of the modules L_j/N_{j-1} are invariants of W .

The following proposition in [EH08] characterizes being anti-nilpotent and having finite length in the category of $R\{F\}$ -modules in terms of Lyubeznik's filtration:

Proposition 4.2 (cf. Proposition 4.8 in [EH08]). *Let the notations and hypothesis be as in Theorem 4.1. Then:*

- (1) *W has finite length as an $R\{F\}$ -module if and only if each of the factors N_j/L_j has finite length in the category of R -modules.*
- (2) *W is anti-nilpotent if and only if in some (equivalently, every) filtration, the nilpotent factors $N_j/L_j = 0$ for every j .*

Remark 4.3. It is worth pointing out that an Artinian R -module W is Noetherian over $R\{F\}$ if and only if in some (equivalently, every) filtration as in Theorem 4.1, each of the factors N_j/L_j is Noetherian as an R -module (same proof as Proposition 4.8 in [EH08]). So W is Noetherian over $R\{F\}$ if and only if it has finite length as an $R\{F\}$ -module. Hence R has FH-finite length if and only if all local cohomology modules $H_{\mathfrak{m}}^i(R)$ are Noetherian $R\{F\}$ -modules.

We also need the following theorem in [BB11] which relates $R\{F\}$ -modules and Cartier modules.

Theorem 4.4 (cf. Proposition 5.2 in [BB11]). *Let (R, \mathfrak{m}) be complete, local and F -finite. Then Matlis duality induces an equivalence of categories between $R\{F\}$ -modules which are*

Artinian as R -modules and Cartier modules which are Noetherian as R -modules. The equivalence preserves nilpotence.

Now we start proving the main theorem of this section. We begin with some lemmas. We will use $^\vee$ to denote the Matlis dual with respect to \mathfrak{m} and $^{\vee_P}$ to denote the Matlis dual with respect to PR_P .

Lemma 4.5. *Let (R, \mathfrak{m}) be a complete local ring. We have $(H_{\mathfrak{m}}^i(R)^\vee)_P^{\vee_P} \cong H_{PR_P}^{i-\dim R/P}(R_P)$.*

Proof. Write $R = T/J$ and $P = Q/J$ for T regular local ring of dimension n . By local duality, we have

$$(H_{\mathfrak{m}}^i(R)^\vee)_P \cong \text{Ext}_T^{n-i}(R, T)_P \cong \text{Ext}_{T_Q}^{n-i}(R_P, T_Q).$$

Now by local duality over R_P ,

$$(H_{\mathfrak{m}}^i(R)^\vee)_P^{\vee_P} \cong H_{PR_P}^{\dim T_Q - (n-i)}(R_P) \cong H_{PR_P}^{i-\dim R/P}(R_P).$$

□

Lemma 4.6. *We have the following:*

- (1) *If M is a nilpotent Cartier module over R , then M_P is a nilpotent Cartier module over R_P*
- (2) *If (M, C) is a simple Cartier module over R with a nontrivial C -action, then (M_P, C_P) is a simple Cartier module over R_P , and if $M_P \neq 0$, then the C_P -action is also nontrivial.*

Proof. (1) is obvious, because if C^e kills M , then C_P^e kills M_P . Now we prove (2). Let N be a Cartier R_P submodule of M_P . Consider the contraction of N in M , call it N' . Then it is easy to check that N' is a Cartier R -submodule of M . So it is either 0 or M because M is simple. But if $N' = 0$ then $N = 0$ and if $N' = M$ then $N = M_P$ because N is an R_P -submodule of M_P . This proves M_P is simple as a Cartier module over R_P . To see the last assertion, notice that if M is a simple Cartier module with a nontrivial C -action, then $C: M \rightarrow M$ must be surjective: otherwise the image would be a proper Cartier submodule. Hence $C_P: M_P \rightarrow M_P$ is also surjective. But we assume $M_P \neq 0$, so C_P is a nontrivial action. □

Theorem 4.7. *Let (R, \mathfrak{m}) be a complete and F -finite local ring. Then the following conditions are equivalent:*

- (1) *R_P is stably FH -finite for every $P \in \text{Spec } R - \{\mathfrak{m}\}$.*
- (2) *R has FH -finite length.*

Proof. By Theorem 4.1, for every $H_{\mathfrak{m}}^i(R)$, $0 \leq i \leq d$, we have a filtration

$$(4.7.1) \quad 0 = L_0 \subseteq N_0 \subseteq L_1 \subseteq N_1 \subseteq \cdots \subseteq L_s \subseteq N_s = H_{\mathfrak{m}}^i(R)$$

of $R\{F\}$ -modules such that every N_j/L_j is F -nilpotent while every L_j/N_{j-1} is simple in the category of $R\{F\}$ -modules, with nontrivial F -action. Now we take the Matlis dual of the above filtration (4.7.1), we have

$$H_{\mathfrak{m}}^i(R)^\vee = N_s^\vee \twoheadrightarrow L_s^\vee \twoheadrightarrow \cdots \twoheadrightarrow N_0^\vee \twoheadrightarrow L_0^\vee = 0$$

such that each $\ker(L_j^\vee \rightarrow N_{j-1}^\vee)$ is Noetherian as an R -module and is simple as a Cartier module with nontrivial C -action, and each $\ker(N_j^\vee \rightarrow L_j^\vee)$ is Noetherian as an R -module and is nilpotent as a Cartier module by Lemma 4.4. When we localize at $P \neq \mathfrak{m}$, we get

$$(H_{\mathfrak{m}}^i(R)^\vee)_P = (N_s^\vee)_P \twoheadrightarrow (L_s^\vee)_P \twoheadrightarrow \cdots \twoheadrightarrow (N_0^\vee)_P \twoheadrightarrow (L_0^\vee)_P = 0$$

with each $\ker((L_j^\vee)_P \twoheadrightarrow (N_{j-1}^\vee)_P)$ a simple Cartier module over R_P whose C_P action is nontrivial if it is nonzero, and each $\ker((N_j^\vee)_P \twoheadrightarrow (L_j^\vee)_P)$ a nilpotent Cartier module over R_P by Lemma 4.6. Now, when we take the Matlis dual over R_P and apply Lemma 4.5, we get a filtration of $R_P\{F\}$ -modules

$$(4.7.2) \quad 0 = L'_0 \subseteq N'_0 \subseteq L'_1 \subseteq \cdots \subseteq L'_s \subseteq N'_s = H_{PR_P}^{i-\dim R/P}(R_P)$$

where $L'_j = (L_j^\vee)_P^{\vee P}$, $N'_j = (N_j^\vee)_P^{\vee P}$, N'_j/L'_j is F -nilpotent and each L'_j/N'_{j-1} is either 0 or simple as an $R_P\{F\}$ -module with nontrivial F -action by Lemma 4.4 again. And we notice that

$$\begin{aligned} N'_j/L'_j &= 0, \forall P \in \operatorname{Spec} R - \{\mathfrak{m}\} \\ \Leftrightarrow (N_j^\vee)_P^{\vee P} / (L_j^\vee)_P^{\vee P} &= 0, \forall P \in \operatorname{Spec} R - \{\mathfrak{m}\} \\ \Leftrightarrow \ker((N_j^\vee)_P \twoheadrightarrow (L_j^\vee)_P) &= 0, \forall P \in \operatorname{Spec} R - \{\mathfrak{m}\} \\ \Leftrightarrow l_R(\ker(N_j^\vee \twoheadrightarrow L_j^\vee)) &< \infty \\ \Leftrightarrow l_R(N_j/L_j) &< \infty \end{aligned}$$

R_P is stably FH-finite for every $P \in \operatorname{Spec} R - \{\mathfrak{m}\}$ if and only if $H_{PR_P}^{i-\dim R/P}(R_P)$ is anti-nilpotent for every $0 \leq i \leq d$ and every $P \in \operatorname{Spec} R - \{\mathfrak{m}\}$. This is because when $0 \leq i \leq d$, $i - \dim R/P$ can take all values between 0 and $\dim R_P$, and if $i - \dim R/P$ is out of this range, then the local cohomology is 0 so it is automatically anti-nilpotent. By Proposition 4.2 (2), this is equivalent to the condition that for every $P \in \operatorname{Spec} R - \{\mathfrak{m}\}$, the corresponding N'_j/L'_j is 0. By the above chain of equivalence relations, this is equivalent to the condition that each N_j/L_j have finite length as an R -module. By Proposition 4.2 (1), this is equivalent to the condition that R have FH-finite length. \square

5. F-PURITY ON THE PUNCTURED SPECTRUM IMPLIES FH-FINITE LENGTH FOR EXCELLENT LOCAL RINGS

In this section we prove that for excellent local rings, F -purity on the punctured spectrum implies FH-finite length. We also prove that the properties such as having FH-finite length, being FH-finite, and being stably FH-finite localize. In fact, if we assume R is complete and F -finite, these results will follow from Theorem 3.8 and Theorem 4.7. Our main point is to make use of the Γ -construction introduced in [HH94a] to prove the general case. We start with a brief review on the Γ -construction. We refer to [HH94a] for details.

Let K be a field of positive characteristic $p > 0$ with a p -base Λ . Let Γ be a fixed cofinite subset of Λ . For $e \in \mathbb{N}$ we denote by $K^{\Gamma, e}$ the purely inseparable field extension of K that is the result of adjoining p^e -th roots of all elements in Γ to K .

Now suppose that (R, \mathfrak{m}) is a complete local ring with $K \subseteq R$ a coefficient field. Let x_1, \dots, x_d be a system of parameters for R , so that R is module-finite over $A = K[[x_1, \dots, x_d]] \subseteq$

R . Let A^Γ denote

$$\bigcup_{e \in \mathbb{N}} K^{\Gamma, e}[[x_1, \dots, x_d]],$$

which is a regular local ring that is faithfully flat and purely inseparable over A . The maximal ideal of A expands to that of A^Γ . Let R^Γ denote $A^\Gamma \otimes_A R$, which is module-finite over the regular ring A^Γ and is faithfully flat and purely inseparable over R . The maximal ideal of R expands to the maximal ideal of R^Γ . The residue field of R^Γ is K^Γ . It is of great importance that R^Γ is F -finite. Moreover, we can preserve some good properties of R if we choose a sufficiently small cofinite subset Γ :

Lemma 5.1 (cf. Lemma 6.13 in [HH94a]). *Let R be a complete local ring. If P is a prime ideal of R then there exists a cofinite set $\Gamma_0 \subseteq \Lambda$ such that $Q = PR^\Gamma$ is a prime ideal in R^Γ for all $\Gamma \subseteq \Gamma_0$.*

Lemma 5.2 (cf. Lemma 2.9 and Lemma 4.3 in [EH08]). *Let R be a complete local ring. Let W be an $R\{F\}$ -module that is Artinian as an R -module such that the F -action is injective. Then for any sufficiently small choice of Γ cofinite in Λ , the action of F on $R^\Gamma \otimes_R W$ is also injective. Moreover, if R^Γ is FH -finite (resp. has FH -finite length), then so is R .*

Now we start proving our main theorems. We first want to show that F -purity is preserved under nice base change. This should be well-known to experts. Since we could not find a good reference, we include a proof here. We will follow the idea in the proof of Theorem 7.24 in [HH94a]. We need the notion of *Frobenius closure*: for any ideal $I \subseteq R$, $I^F = \{x \in R \mid x^{p^e} \in I^{[p^e]} \text{ for some } e\}$. If R is F -pure, then every ideal is Frobenius closed. We will see that under mild conditions on the ring, the converse also holds.

We also need the notion of *approximately Gorenstein ring* introduced in [Hoc77]: (R, \mathfrak{m}) is approximately Gorenstein if there exists a decreasing sequence of \mathfrak{m} -primary ideals $\{I_t\}$ such that every R/I_t is a Gorenstein ring and the $\{I_t\}$ are cofinal with the powers of \mathfrak{m} . That is, for every $N > 0$, $I_t \subseteq \mathfrak{m}^N$ for all $t \gg 1$. We will call such a sequence of ideals an *approximating sequence of ideals*. Note that for an \mathfrak{m} -primary ideal I , R/I is Gorenstein if and only if I is an irreducible ideal, i.e. it is not the intersection of two strictly larger ideals. Every reduced excellent local ring is approximately Gorenstein (see [Hoc77]).

Lemma 5.3. *Let (R, \mathfrak{m}) be an excellent local ring. Then R is F -pure if and only if there exists an approximating sequence of ideals $\{I_t\}$ such that $I_t^F = I_t$.*

Proof. If (R, \mathfrak{m}) is excellent and F -pure then R is reduced, hence approximately Gorenstein. So there exists an approximating sequence of ideals $\{I_t\}$. $I_t^F = I_t$ follows because R is F -pure. For the other direction, we use $R^{(1)}$ to denote the target ring under the Frobenius endomorphism $R \rightarrow R^{(1)}$ and we want to show $R \rightarrow R^{(1)}$ is pure. It suffices to show that $E_R \hookrightarrow R^{(1)} \otimes_R E_R$ is injective where E_R denotes the injective hull of the residue field of R . But it is easy to check that $E_R = \varinjlim_t \frac{R}{I_t}$. Hence $E_R \hookrightarrow R^{(1)} \otimes_R E_R$ is injective if $\frac{R}{I_t} \hookrightarrow \frac{R^{(1)}}{I_t R^{(1)}}$ is injective for all t . But this is true because $I_t^F = I_t$. \square

Proposition 5.4. *Let $(R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ be a faithfully flat extension of excellent local rings such that the closed fibre $S/\mathfrak{m}S$ is Gorenstein and F -pure. If R is F -pure, then S is F -pure.*

Proof. Let $\{I_k\}$ be an approximating sequence of ideals in R . Let x_1, \dots, x_n be elements in S such that their image form a system of parameters in $S/\mathfrak{m}S$. Since $S/\mathfrak{m}S$ is Gorenstein, (x_1^t, \dots, x_n^t) is an approximating sequence of ideals in $S/\mathfrak{m}S$. So that $I_k + (x_1^t, \dots, x_n^t)$ is an approximating sequence of ideals in S (see the proof of Theorem 7.24 in [HH94a]).

To show that S is F -pure, it suffices to show every $I_k + (x_1^t, \dots, x_n^t)$ is Frobenius closed by Lemma 5.3. Therefore we reduce to showing that if I is an irreducible ideal primary to \mathfrak{m} in R and x_1, \dots, x_n are elements in S such that their image form a system of parameters in $S/\mathfrak{m}S$, then $(IS + (x_1, \dots, x_n))^F = IS + (x_1, \dots, x_n)$ in S .

Let v and w be socle representatives of R/I and $(S/\mathfrak{m}S)/(x_1, \dots, x_n)(S/\mathfrak{m}S)$ respectively. It suffices to show $vw \notin (IS + (x_1, \dots, x_n))^F$. Suppose we have

$$v^q w^q \in I^{[q]}S + (x_1^q, \dots, x_n^q).$$

Taking their image in $S/(x_1^q, \dots, x_n^q)$, we have

$$\overline{v^q w^q} \in \overline{I^{[q]}S},$$

therefore

$$\overline{w^q} \in (\overline{I^{[q]}S} : \overline{v^q}) = (I^{[q]} : v^q)(S/(x_1^q, \dots, x_n^q)S)$$

where the second equality is because $S/(x_1^q, \dots, x_n^q)S$ is faithfully flat over R . So we have

$$w^q \in (I^{[q]} : v^q)S + (x_1^q, \dots, x_n^q).$$

Since R is F -pure, $(I^{[q]} : v^q) \in \mathfrak{m}$. So after taking image in $S/\mathfrak{m}S$, we get that

$$w^q \in (x_1^q, \dots, x_n^q)(S/\mathfrak{m}S).$$

Hence $w \in (x_1^q, \dots, x_n^q)^F$ in $S/\mathfrak{m}S$, this contradict to the condition that $S/\mathfrak{m}S$ be F -pure. \square

Next we want to show the openness of F -pure locus when R is F -finite, where the F -pure locus is defined as the set $\{P \in \text{Spec } R \mid R_P \text{ is } F\text{-pure}\}$.

Lemma 5.5. *Let (R, \mathfrak{m}) be an F -finite local ring. Then the F -pure locus is open in $\text{Spec } R$ in the Zariski topology.*

Proof. We will use $R^{(1)}$ to denote the target ring under the Frobenius endomorphism $R \rightarrow R^{(1)}$. Since R is F -finite, $R^{(1)}$ is finitely generated as an R -module.

It suffice to show that if R_P is F -pure, then there exists $x \notin P$ such that R_x is F -pure. Since R is F -finite, so is R_P . Therefore R_P is F -split. That is, the map $R_P \hookrightarrow R_P^{(1)}$ splits as R -modules. This implies that the map

$$\text{Hom}_{R_P}(R_P^{(1)}, R_P) \rightarrow \text{Hom}_{R_P}(R_P, R_P)$$

is surjective. Since $R^{(1)}$ is finitely generated as an R -module, $\text{Hom}_{R_P}(R_P^{(1)}, R_P) \cong \text{Hom}_R(R^{(1)}, R)_P$. So we know that

$$\text{Hom}_R(R^{(1)}, R)_P \rightarrow \text{Hom}_R(R, R)_P$$

is surjective. Hence we may pick $x \notin P$ such that

$$\text{Hom}_R(R^{(1)}, R)_x \rightarrow \text{Hom}_R(R, R)_x$$

is surjective. This proves that R_x is F -split, hence F -pure. \square

Now we show that F -purity on the punctured spectrum is preserved under the Γ -construction when we pick Γ sufficiently small and cofinite in Λ .

Proposition 5.6. *Let (R, \mathfrak{m}) be a complete local ring such that R_P is F -pure on the punctured spectrum $\text{Spec } R - \{\mathfrak{m}\}$. Then for any sufficiently small choice of Γ cofinite in Λ , R^Γ is F -pure on the punctured spectrum $\text{Spec } R^\Gamma - \{\mathfrak{m}R^\Gamma\}$.*

Proof. Because R^Γ is purely inseparable over R , for all $P \in \text{Spec } R$ there is a unique prime ideal $P^\Gamma \in \text{Spec } R^\Gamma$ lying over P . In particular, $\text{Spec } R^\Gamma \cong \text{Spec } R$. Since R^Γ is F -finite, we know the F -pure locus of each R^Γ , call it X_Γ , is open in $\text{Spec } R^\Gamma = \text{Spec } R$ by Lemma 5.5. Since open sets in $\text{Spec } R$ satisfy ACC, we know that there exists Γ such that X_Γ is maximal. We will show that $X_\Gamma \supseteq \text{Spec } R - \{\mathfrak{m}\}$. This will prove R^Γ is F -pure on the punctured spectrum $\text{Spec } R^\Gamma - \{\mathfrak{m}R^\Gamma\}$.

Suppose there exists $Q \neq \mathfrak{m}$ such that $Q \notin X_\Gamma$. We may pick $\Gamma' \subseteq \Gamma$ sufficiently small and cofinite in Λ such that $QR^{\Gamma'}$ is prime (that is, $QR^{\Gamma'} = Q^{\Gamma'}$) by Lemma 5.1. So $R_Q \rightarrow R_{Q^{\Gamma'}}^{\Gamma'}$ is faithfully flat whose closed fibre is a field. By Proposition 5.4, $R_{Q^{\Gamma'}}^{\Gamma'}$ is F -pure. Since $\Gamma' \subseteq \Gamma$, $R^{\Gamma'} \rightarrow R^\Gamma$ is faithfully flat so $R_{P^{\Gamma'}}^{\Gamma'} \rightarrow R_{P^\Gamma}^\Gamma$ is faithfully flat for each $P \in \text{Spec } R$. Now for $P \in X_\Gamma$, $R_{P^\Gamma}^\Gamma$ is F -pure, hence so is $R_{P^{\Gamma'}}^{\Gamma'}$. So $X_{\Gamma'} \supseteq X_\Gamma \cup \{Q\}$, which is a contradiction since we assume that X_Γ is maximal. \square

Theorem 5.7. *Let (R, \mathfrak{m}) be an excellent local ring such that R_P is F -pure for every $P \in \text{Spec } R - \{\mathfrak{m}\}$. Then R has FH-finite length.*

Proof. We look at the chain of faithfully flat ring extensions:

$$R \rightarrow \widehat{R} \rightarrow \widehat{R}^\Gamma \rightarrow \widehat{\widehat{R}^\Gamma}.$$

Since R is excellent, for every $Q_0 \in \text{Spec } \widehat{R} - \{\mathfrak{m}\widehat{R}\}$ lying over P in R , $R_P \rightarrow \widehat{R}_{Q_0}$ has geometrically regular fibres, so \widehat{R}_{Q_0} is F -pure by Proposition 5.4. Hence \widehat{R} is F -pure on the punctured spectrum. So by Proposition 5.6, we can pick Γ sufficiently small and cofinite in Λ such that \widehat{R}^Γ is F -pure on the punctured spectrum.

For every $Q \in \text{Spec } \widehat{R}^\Gamma - \{\mathfrak{m}\widehat{R}^\Gamma\}$, let $Q_1 \neq \mathfrak{m}\widehat{R}^\Gamma$ be the contraction of Q to \widehat{R}^Γ . Since $\widehat{R}_{Q_1}^\Gamma$ is F -finite, it is excellent by [Kun76]. So the closed fibre of $\widehat{R}_{Q_1}^\Gamma \rightarrow \widehat{\widehat{R}^\Gamma}_Q$ is geometrically regular. So $\widehat{\widehat{R}^\Gamma}_Q$ is F -pure by Proposition 5.4.

It follows that, for sufficiently small choice of Γ cofinite in Λ , $\widehat{\widehat{R}^\Gamma}_Q$ is F -pure for every $Q \in \text{Spec } \widehat{R}^\Gamma - \{\mathfrak{m}\widehat{R}^\Gamma\}$. Now by Theorem 3.8 applied to $\widehat{\widehat{R}^\Gamma}_Q$ and Theorem 4.7 applied to $\widehat{\widehat{R}^\Gamma}$, we know $\widehat{\widehat{R}^\Gamma}$ has FH-finite length. Hence so does R by Lemma 2.6 and Lemma 5.2. \square

Lemma 5.8. *Let R be a complete local ring. Let W be a simple $R\{F\}$ -module that is Artinian over R with nontrivial F -action. Then for any sufficiently small choice of Γ cofinite in Λ , $W \otimes_R R^\Gamma$ is a simple $R^\Gamma\{F\}$ -module with nontrivial F -action.*

Proof. By Lemma 5.2, we may pick Γ sufficiently small and cofinite in Λ such that F acts injectively on $W \otimes_R R^\Gamma$. I claim such a $W \otimes_R R^\Gamma$ must be a simple $R^\Gamma\{F\}$ -module. If not, then by Theorem 4.1, we have $0 \subsetneq L \subsetneq W \otimes_R R^\Gamma$ where L is simple with nontrivial Frobenius action. Now we pick $0 \neq x \in L$. Because R^Γ is purely inseparable over R , there exists e such that $0 \neq F^e(x) \in W$. Hence $L \cap W \neq 0$. But it is straightforward to check that $L \cap W$ is an $R\{F\}$ -submodule of W . So $L \cap W = W$ since W is simple. Hence $L \supseteq W \otimes_R R^\Gamma$ which is a contradiction. \square

Proposition 5.9. *Let (R, \mathfrak{m}) be a complete local ring. Then*

- (1) *If R has FH-finite length, then so does R^Γ for Γ sufficiently small and cofinite in Λ .*
- (2) *If R is stably FH-finite, then so is R^Γ for Γ sufficiently small and cofinite in Λ .*

Proof. By Theorem 4.1, for every $0 \leq i \leq d = \dim R$, we have a filtration of $R\{F\}$ -modules

$$0 = L_0 \subseteq N_0 \subseteq L_1 \subseteq N_1 \subseteq \cdots \subseteq L_s \subseteq N_s = H_{\mathfrak{m}}^i(R)$$

with each N_j/L_j F -nilpotent and L_j/L_{j-1} simple as an $R\{F\}$ -module with nonzero F -action. By Lemma 5.8, we can pick Γ sufficiently small and cofinite in Λ such that, for all i , all $L_j/L_{j-1} \otimes_R R^\Gamma$ are simple with nonzero F -action. Hence

$$0 = L_0^\Gamma \subseteq N_0^\Gamma \subseteq L_1^\Gamma \subseteq N_1^\Gamma \subseteq \cdots \subseteq L_s^\Gamma \subseteq N_s^\Gamma = H_{\mathfrak{m}}^i(R^\Gamma)$$

where $L_j^\Gamma = L_j \otimes_R R^\Gamma$ and $N_j^\Gamma = N_j \otimes_R R^\Gamma$ is a corresponding filtration of $H_{\mathfrak{m}}^i(R^\Gamma)$. Now both (1) and (2) are clear from Proposition 4.2. \square

Theorem 5.10. *Let (R, \mathfrak{m}) be a local ring that has FH-finite length (resp. is FH-finite or stably FH-finite). Then the same holds for R_P for every $P \in \operatorname{Spec} R$.*

Proof. It suffices to show that if (R, \mathfrak{m}) has FH-finite length, then R_P is stably FH-finite for every $P \neq \mathfrak{m}$. We first notice that \widehat{R} has FH-finite length by Lemma 2.6. We pick Γ sufficiently small and cofinite in Λ such that \widehat{R}^Γ still has FH-finite length by Proposition 5.9. Now we complete again, and we get that $B = \widehat{\widehat{R}^\Gamma}$ is an F -finite complete local ring that has FH-finite length by Lemma 2.6, and the maximal ideal in B is $\mathfrak{m}B$. Notice that $R \rightarrow B$ is faithfully flat, hence for every $P \neq \mathfrak{m}$, we may pick $Q \in \operatorname{Spec} B - \{\mathfrak{m}B\}$ such that $R_P \rightarrow B_Q$ is faithfully flat (in particular, pure) and PB_Q is primary to QB_Q . So $\widehat{R_P} \rightarrow \widehat{B_Q}$ is split. By Theorem 4.7 applied to B , B_Q is stably FH-finite. Hence so is $\widehat{B_Q}$ by Lemma 2.6. Now we apply Corollary 3.5, we see that $\widehat{R_P}$ is stably FH-finite. Hence so is R_P by Lemma 2.6. \square

6. SOME EXAMPLES

Since stably FH-finite trivially implies F -injective, it is quite natural to ask whether FH-finite implies F -injective. The following example studied in [EH08] shows this does not hold in general.

Example 6.1 (*cf.* Example 2.15 in [EH08]). Let $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$ where K is a field of characteristic different from 3. This is a Gorenstein ring of dimension 2. And it can be checked that the only nontrivial F -compatible submodule in $H_{\mathfrak{m}}^2(R)$ is its socle, a copy of K . Hence R is FH-finite. But it is known that R is F -pure (equivalently, F -injective since R is Gorenstein) if and only if the characteristic of K is congruent to 1 mod 3. Hence if the characteristic is congruent to 2 mod 3, we get an example of FH-finite ring which is not F -injective.

Another natural question to ask is whether the converse of Theorem 3.8 is true. The next example will show this is also false in general. We recall a theorem in [Smi97]:

Theorem 6.2 (*cf.* Theorem 2.6 in [Smi97] or Proposition 2.12 in [EH08]). *Let (R, \mathfrak{m}) be an excellent Cohen-Macaulay local ring of dimension d . Then R is F -rational if and only if $H_{\mathfrak{m}}^d(R)$ is a simple $R\{F\}$ -module.*

Corollary 6.3. *Let (R, \mathfrak{m}) be an excellent F -rational local ring of dimension d . Then R is stably FH-finite.*

Proof. This follows immediately from Theorem 6.2 and Theorem 2.3 because when $H_{\mathfrak{m}}^d(R)$ is a simple $R\{F\}$ -module, it is obviously anti-nilpotent. \square

Example 6.4 (*cf.* Example 7.15 in [HH94b]). Let $R = K[t, xt^4, x^{-1}t^4, (x+1)^{-1}t^4]_{\mathfrak{m}} \subseteq K(x, t)$ where $\mathfrak{m} = (t, xt^4, x^{-1}t^4, (x+1)^{-1}t^4)$. Then R is F -rational but not F -pure. Hence by our Corollary 6.3, R is a stably FH-finite Cohen-Macaulay ring that is not F -pure.

Also note that even when R is Cohen-Macaulay and F -injective, it is not always FH-finite, and does not even always have FH-finite length. We have the following example:

Example 6.5 (*cf.* Example 2.16 in [EH08]). Let k be an infinite perfect field of characteristic $p > 2$, $K = k(u, v)$, where u and v are indeterminates, and let $L = K[y]/(y^{2p} + uy^p - v)$. Let $R = K + xL[[x]] \subseteq L[[x]]$. Then R is a complete F -injective Cohen-Macaulay domain of dimension 1 which is not FH-finite. Notice that by Theorem 2.3, $R[[x]]$ is an F -injective Cohen-Macaulay domain of dimension 2 that does not have FH-finite length (this was not pointed out in [EH08]).

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